OPTIMAL EVASIVE TRAJECTORIES OF AN ISOTROPIC ACOUSTIC RADIATOR

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THESIS

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AN ISOTROPIC ACOUSTIC RADIATOR

by

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of
an Isotropic Acoustic Radiator

by

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1.0 INTRODUCTION

1.1 THE PROBLEM UNDER CONSIDERATION

It is the nature of submarine warfare to avoid detection. Indeed, one of the major advantages of the submarine is surprise. The passive detection of low level acoustic signals (in a water environment) is a function of many factors. Among these are the power density of the signal at the detector and the period of time over which the signal is received. This paper seeks to examine the differences in suggested evasion tactics which result from a study of these two factors.

The power density received by the detector may be represented as shown in Figure 1.1.1.

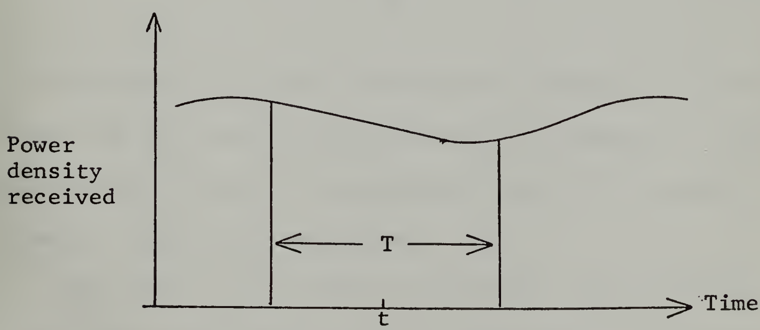


Figure 1.1.1

Due to the nature of the detection equipment, a memory, or integration time is required. The integration time is shown as T in the figure. For an evader, which is producing the noise signal and wishes to minimize the probability of detection (all other factors being held constant), the objective is to minimize the maximum area over time under the power density curve over the integration time. The memory requirement of the equipment is the same for each instant of time. While time advances this "window" of integration time, T, slides along with the power density curve. Expressed mathematically the objective is to:

$$\underset{P}{\text{Min}} \quad \underset{t}{\text{Max}} \int_{t-\frac{T}{2}}^{t+\frac{T}{2}} P(u) \quad du \quad . \tag{1.1.1}$$

A solution in the general case is not proposed, only very large and very small values of T will be examined. That is, the extreme effects of integration time on the optimal evasive trajectory are studied.

Typically for an evader at 60 miles range initially and a relative closing velocity of 20 knots an engagement with a searcher will be approximately three hours.

Integration times of less than an hour are available with today's equipment. As an upper bound the case where the integration time is much greater than the engagement time may essentially be called infinite. The lower bound would be zero. The boundary cases are then "infinite" and "zero" integration time. The goal is to solve for the two optimal evasive trajectories and to compare them.

The infinite integration time problem can be formulated to fit within the general framework of the control problem. This class of problems is an outgrowth of the calculus of variations. It is one objective of this paper to apply the maximum principle to the infinite integration time problem. As the integration time reduces to zero, the objective becomes to minimize the peak power. This problem is formulated in ordinary differential equations.

1.2 NOISE POWER RADIATION AND ATTENUATION

The noise power radiation, for a line spectrum of a submarine has been modeled to have the mathematical form

$$P(v) = a + b\ v^4 \qquad (1.2.1)$$

where; a, b are constant coefficients to be determined, and v is velocity, measured in knots.

Data was obtained on the radiated noise levels of submarines as a function of velocity at a particular line frequency. A least squares curve fit was made to the data. The form of equation (1.2.1) was used. A correlation coefficient, R^2, greater than .9 was obtained. This was considered sufficient justification for the use of equation (1.2.1) as a model. The power equation was then normalized to the form,

$$P(v) = 1.0 + b/a\ v^4. \qquad (1.2.2)$$

The original data used for the development of the coefficients is classified. Since the units in which power is is measured are unimportant as far as the analysis is concerned, and so that this paper could be unclassified, the coefficients were normalized. The final form of the power equation used was,

$$P(v) = 1.0 + 0.00005\, v^4. \qquad (1.2.3)$$

A further assumption is that the submarine is an isotropic radiator. That is, the effects of lobe radiation are not considered. The attenuation of the noise power is also assumed to follow an inverse square law. This was strictly for convenience, since the solution can easily accomodate any attenuation function as long as it is a function of range. The simplifying assumption about isotropic radiation is more serious, and as yet, no manner of incorporation into the proposed solution has been discovered.

1.3 HYPOTHETICAL SUBMARINE ENCOUNTER

Suppose submarine E, the evader, is assigned a patrol area in which she is motivated to remain, but may maneuver at any course and speed she chooses. Suppose further that submarine S is searching along a straight path with a constant search velocity V_s for E. Assume also that E is directly on the search path of S as in Figure 1.3.1.

Figure 1.3.1

In order to attempt to avoid detection, E must detect S and then move away. The situation where E detects S first is only considered here. Once S detects E, the game is over. The hope is that E will know about the presence of S, and will be able to maneuver in such a manner that minimizes her probability of detection. The evader is not constrained in this problem by acceleration, and may therefore take on any initial velocity desired. The question is then, what is the optimal trajectory which minimizes the probability of detection?

The nature of the solution to this problem gives the answer to a set of possible initial evasion states. It is not necessary that the evader start from a point directly on the path of the searcher. Evasion may begin at any state position.

2.0 PROBLEM FORMULATION

2.1 INFINITE INTEGRATION TIME FORMULATION

As an extreme the integration time of the searcher's equipment could be infinity. That is, for this situation the objective of the evader would be to minimize total noise energy received by the searcher over time. The total energy received by the searcher can be expressed as an integral:

$$\text{total energy received} = J = \int_0^{\infty} \text{Power received } (t) \; dt. \tag{2.1.1}$$

The power received by the searcher is a function of the evader's velocity and distance. Assuming an inverse square law,

$$\text{Power Received} = \frac{P(v)}{R^2} . \tag{2.1.2}$$

The problem may now be expressed as,

$$\text{Min } J = \int_0^{\infty} \frac{P(v)}{R^2} \, dt \quad , \tag{2.1.3}$$

where v and R are both functions of time.

This class of problems, which results from the calculus of variations, may be solved by one of two modern approaches to the control problem. (2)

Older methods of static optimization involved the introduction of Lagrange multipliers, one for each constraint; defining a Lagrangian expression and finding a saddle point of this expression.

One modern approach to the control problem is dynamic programing and the application of the principle of optimality, which states that:

> "An optimal policy has the property that, whatever the initial state and decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision." (3)

The dynamic programming approach basically assumes an optimal performance function exists and defines a recurrance relationship. The partial derivative of this recurrance relationship is called Bellman's equation, the solution being the optimal performance function sought.

Another modern approach is the maximum principle. (2) The maximum principle can be considered as an extension of the method of Lagrange multipliers to dynamic optimization

problems. The general control problem to be solved by the maximum principle is;

$$\text{Max } J = \int_{t_0}^{t_1} I(x,u,t)\,dt + F(x_1,t_1)$$

$$\text{subject to:} \quad \frac{dx}{dt} = f(x,u,t)$$

$$x(t_0) = x_0$$

$$x(t_1) = x_1 \qquad (2.1.4)$$

$$\left[u(t)\right] \in U$$

where I( ), F( ), and f( ) are continuously differentiable functions; t_0, x_0 are parameters defining the terminal surface; and $\left[u(t)\right]$, the control trajectory must belong to the given control set U.

In control theory there are state variables, control variables, and costate variables. In this problem the state variables, here represented by x are actually the x and y position coordinates, in relative space. The variables under "control" are here represented by u, but in the actual problem are velocity (v) and heading (ϕ). Velocity and course heading are both functions of time.

For each constraint a "costate" variable is introduced. This costate variable, $\lambda(t)$, is the dynamic equivalent of a Lagrange multiplier. This leads to the development of the Hamiltonian function which is defined to be;

$$H(x,u,t) = I(x,u,t) + \quad f(x,u,t). \qquad (2.1.5)$$

The infinite integration time problem may now be expressed as;

$$\text{Max } J = \int_{0}^{\infty} \frac{-P(v)}{x^2 + y^2} \, dt$$

$$\text{subject to;} \quad \frac{dx}{dt} = -V_s + v \sin \phi$$

$$\frac{dy}{dt} = v \cos \phi \qquad (2.1.6)$$

$$\left.\begin{matrix} x(t_0) \\ y(t_0) \end{matrix}\right\} \text{ initial conditions}$$

$$\left.\begin{matrix} x(t_1) \\ y(t_1) \end{matrix}\right\} \text{ terminal conditions}$$

where, ϕ is a function of time and is the course heading, and v, also a function of time, is the velocity. In order to achieve a practical mathematical solution, infinity is approximated with a large number.

The constraints dx/dt and dy/dt are obtained from the geometry of Figure 3.1.1. Note that the course heading, ϕ, is measured from the vertical.

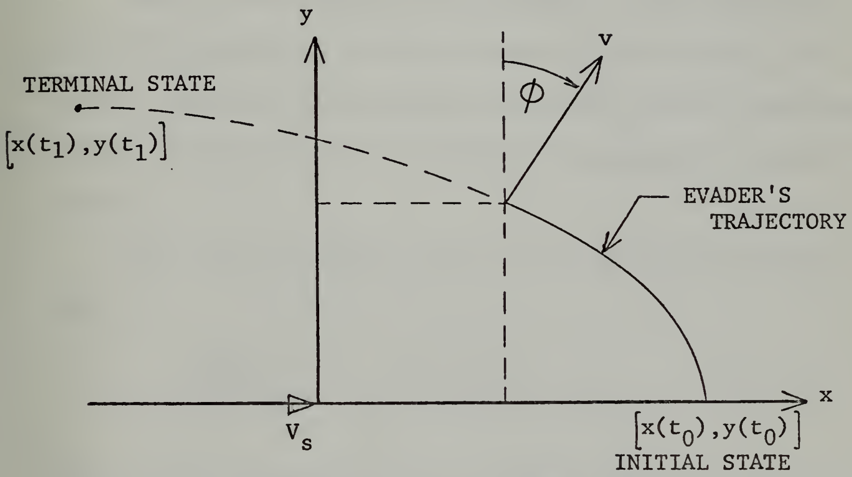


Figure 2.1.1 - geometry relative to the searcher

In this problem the correspondence of functions is;

$$I(\ \ ) = \frac{-P(v)}{x^2 + y^2} \tag{2.1.7}$$

$$F(\ \ ) = 0 \tag{2.1.8}$$

$$\frac{dx}{dt} = -V_s + v \sin\phi \tag{2.1.9}$$

$$\frac{dy}{dt} = v \cos \phi \qquad (2.1.10)$$

The solution to this problem is contained in appendix A.

Just as the Lagrange multipliers of static problems yield information on the sensitivity of the solution, the costate variables of the maximum principle yield information on the sensitivity of the solution to variations in parameters.

Let J* be the optimal value of the objective function then

$$\frac{\partial J^*}{\partial x(t_0)} = \lambda^* (t_0) \quad . \qquad (2.1.11)$$

That is, the sensitivity of the optimal objective function to changes in the initial state variables is given by the corresponding initial costate variables. If a costate variable is small then the relative sensitivity of the optimal objective function will be small for small changes of the initial state variable.

In static problems the Lagrange multipliers have the interpretation of "shadow price" and a dynamic analogue exists.

2.1 ZERO INTEGRATION TIME FORMULATION AND SOLUTION

As the integration time is reduced to zero, it can be argued that the objective of the evader is to minimize the peak power received by the searcher. The problem may be viewed as one in which there exists a threshold of power density level, which, if exceeded, would result in detection by the searcher.

Assume there exists a set of state positions S where, for a given threshold of detection, the evader could not escape. Assume also there is another set S where the evader could always escape. The sets could take the form shown in Figure 2.2.1.

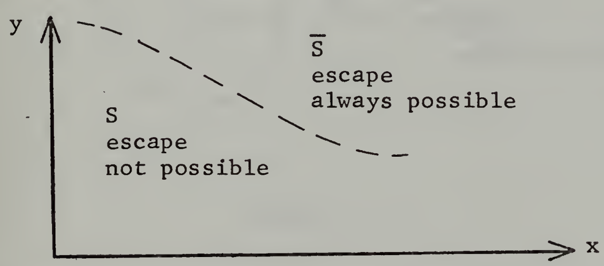


Figure 2.2.1

There must exist some boundary between the two sets of state positions. It would be of interest to know the general shape of this boundary.

Suppose the evader is in some state as shown in Figure 2.2.2. For a given power density threshold K there will be associated a corresponding velocity v_k that satisfies,

$$\frac{P(v_k)}{x^2 + y^2} = K \tag{2.2.1}$$

where v_k is the maximum velocity the evader can have without exceeding the threshold of detection. Actually the evader is allowed to choose any velocity and heading within the circle of Figure 2.2.2.

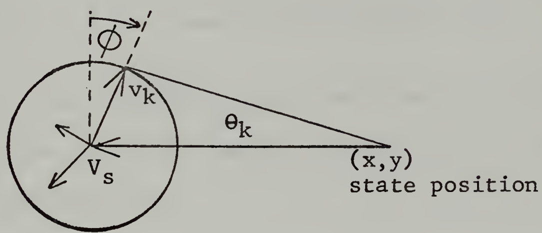


Figure 2.2.2

The maximum velocity possible and the state position determine the angle θ_k. That is,

$$\theta_k = \text{arc sin } (v_k/V_s) \tag{2.2.2}$$

By geometry $\phi = \theta_k$. The question the evader must answer is, which controls should be chosen.

Suppose the evader is in set S, the set of states where he cannot escape detection no matter which controls are chosen. In Figure 2.2.3 it can be seen that the shape of the boundary must be such that its slope at the state position is greater than or equal to θ_k.

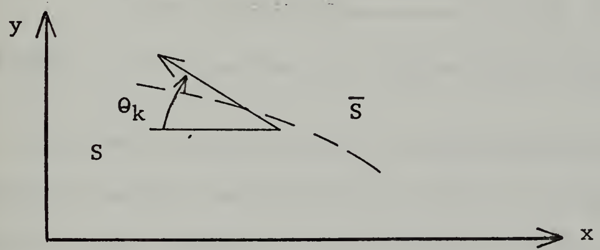


Figure 2.2.3 - boundary with slope $< \theta_k$

If the slope of the boundary were less than θ_k it would be possible for the evader to escape from the set S. This would be a contradiction.

Suppose the evader is in the set of states where he can always escape, provided he chooses the right control. Since the evader is not allowed to choose a control which produces a θ greater than θ_k, the slope of the boundary must be less than or equal to θ_k. Otherwise E would always move from the region of escape to non-escape as shown in Figure 2.2.4.

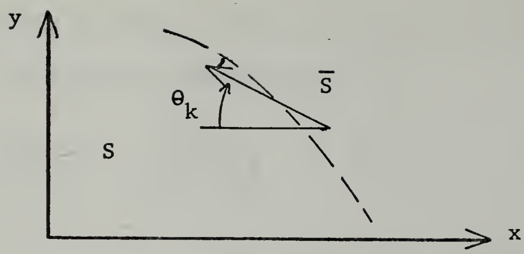


Figure 2.2.4 - boundary with slope greater than θ_k

In order to resolve these contradictions, the evader should choose the controls which produce a θ_k which is tangent to the boundary. This is shown in Figure 2.2.5.

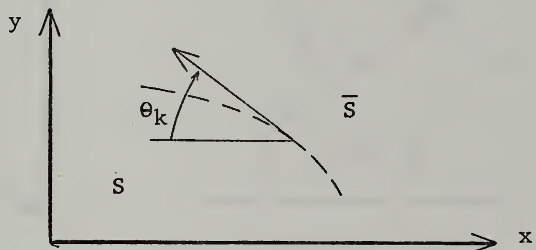


Figure 2.2.5 - boundary with slope equal to θ_k

As shown by Figure 2.2.2 this θ_k is the result of choosing the maximum velocity v_k.

Since the choice of a velocity control will be always to maximize velocity for a given power level which means to always produce as much power as possible, the result is that the power level will remain constant at the detector.

For the zero integration time situation the objective is to maintain a constant power level received by the searcher. This may be expressed as

$$\mathrm{PFINAL} = \frac{P(v_f)}{x_f^2 + y_f^2} = K \qquad (2.2.3)$$

where K is a constant, and the f subscript indicates the final value at the termination of the trajectory.

The equations of motion may now be derived from the geometry as shown in Figure 2.2.6.

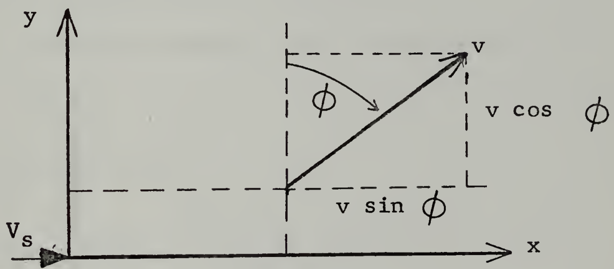


Figure 2.2.6

Velocity, v, and heading, ϕ, are functions of time. The equations of motion are;

$$\frac{dx}{dt} = -V_s + v \sin\phi \qquad (2.2.4)$$

$$\frac{dy}{dt} = v \cos\phi \qquad (2.2.5)$$

Equations (2.2.4) and (2.2.5) may be combined to express

$$\frac{dy}{dx} = \frac{v \cos \phi}{- V_s + v \sin \phi} \tag{2.2.6}$$

The problem now arises that the initial conditions which would allow a solution to the equations of motion are part of the results that are sought. One question that the evader would have, is what initial velocity and heading should be chosen. Further analysis of the nature of the trajectory should supply a solution.

The following remarks refer to Figure 2.2.7.

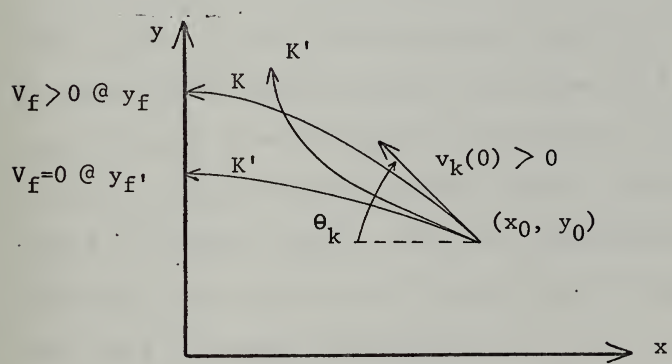


Figure 2.2.7

Suppose the evader is in state (x_0,y_0) and has a given power level, K. This implies that $v_k(0)$ is greater than zero, where $v_k(0)$ is the velocity at state (x_0,y_0). There

[illegible]

[illegible]

[illegible]

[illegible]

will be associated with this power level, K, a trajectory, not necessarily optimal.

Suppose the final velocity at x=0 is greater than zero. The game is considered over at x=0 since this would be the minimum range, and hence, the evader could simply stop all action and continue to improve his situation. It is contended that there exists a K' less than K, since a $v_{k'}(0)$ could be chosen which is less than $v_k(0)$. This would imply that

$$\frac{P\ (v_{k'}(0))}{x_0^2 + y_0^2} < \frac{P\ (v_k(0))}{x_0^2 + y_0^2} . \qquad (2.2.7)$$

Since $v_{k'}(0)\ \ v_k(0)$, this implies that $\theta_{k'} < \theta_k$. Initially, at least, the K' trajectory will lie below the K trajectory. Could K' cross K as indicated in the figure? If this were to occur, the $\theta_{k'}$ would be greater than θ_k, which would imply that K' is greater than K. This is a contradiction, and therefore, the trajectory for power level K' cannot cross that for K, but must lie below it.

Since trajectory K' is below that for K, the terminating distance $y_{f'}$ must be less than y_f. From equation (2.2.3) the conclusion is that $v_{f'}$ must be less than v_f. That is, the terminal velocity must decrease. The result then is to

choose a lower power level until the terminal velocity is zero.

Could the terminal velocity go to zero before reaching the y axis? This is not a feasible result, since this is not the state position of minimum range. Due to the movement of the searcher the range would decrease and the power received would increase above the detection threshold.

The object was to discover the initial conditions so the equations of motion could be solved. Instead the termination conditions are known. These will be used as initial conditions by reversing time. The system of equations to be solved is now;

$$\frac{dx}{dt} = V_s - v \sin \phi \qquad (2.2.8)$$

$$\frac{dy}{dt} = - v \cos \phi \qquad (2.2.9)$$

and,

$$x(t_0) = 0$$

$$y(t_0) = \text{parameter}$$

$$v(t_0) = 0$$

$$\phi(t_0) = 0$$

are the initial conditions.

3.0 RESULTS

The results of the analysis consist of giving advice on the choice of control variables to an evader. Both cases are presented; infinite integration time in graph (1.A.) and zero integration time in graph (1.B.) Advice is given in both cases only for the first quadrant, as this is the region of greatest concern. Four evasive trajectories were calculated and are shown on each graph. Those trajectories considered were for initial evasion ranges of 60, 40, 30, and 20 miles. Trajectories were calculated for each of the four initial ranges both for infinite and zero integration time. The set of graphs labeled 2.-- are for a search velocity of 15 knots. Set 3.-- is a replication of set 2.-- but for a search velocity of 10 knots.

For the 10 knot search velocity and infinite integration time trajectories were calculated for initial evasion ranges of 30 and 20 miles. The 60 and 40 mile curves went off scale and were omitted.

Control variable measurements were taken from the velocity and heading graphs and these control vectors were plotted on the relative movement plots in graphs 1.A and

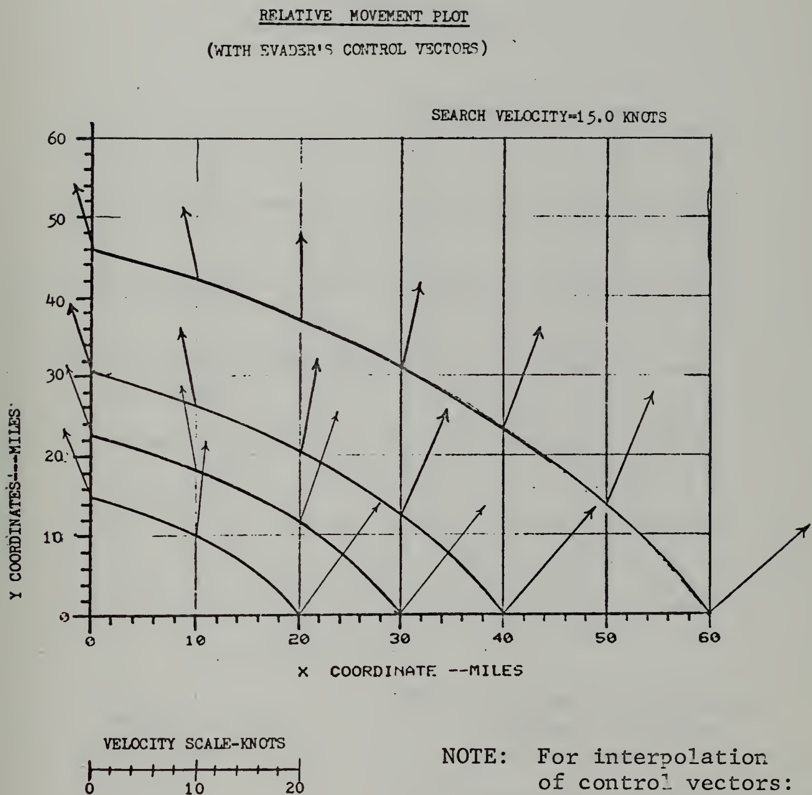
1.B. These vectors are the advice which would be given to an evader. It is not necessary that an evader start at a state position such that he is directly on the path of the searcher. He may start at any position and begin evasion with the control vector belonging to that state. If the searcher should change speed it would be necessary for the evader to consult a new set of control variables and continue the evasion.

Graph 1.C shows a comparison of the relative movement plots of infinite and zero integration times. Only the 60 and 30 mile curves are presented for clarity. The control vectors for the two situations do not appear significantly different during the initial stages of the trajectories, and the result is similar trajectories. As the range decreases the effects of integration time become more apparent. The graph shows that near the minimum ranges the effects of integration time are critical and hence it would be of interest to know what the integration time of the detector is. During the initial stages of evasion knowledge of the integration time is not critical.

Graph 1.D is a representative plot of the costate variables. The costate variables, in part, represent the importance of their corresponding state variable. Costate

variable λ_1 corresponds to the x coordinate and λ_2 to the y coordinate. The graph shows that the evader, starting at state position (x = 60, y = 0) wants to increase y and values y slightly over x. That is, the evader values an increase in the lateral distance, y over an increase in time. This is true throughout the trajectory.

INFINITE INTEGRATION TIME



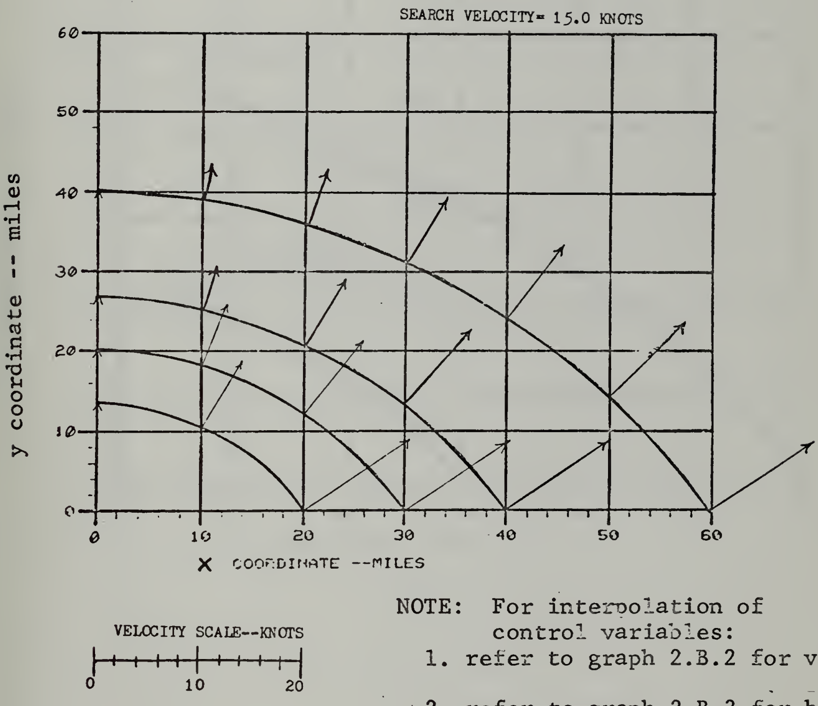
NOTE: For interpolation of control vectors:
1. refer to graph 2.A.2 for velocity
2. refer to graph 2.A.3 for heading

GRAPH 1.A

ZERO INTEGRATION TIME

RELATIVE MOVEMENT PLOT

(WITH EVADER'S CONTROL VECTORS)

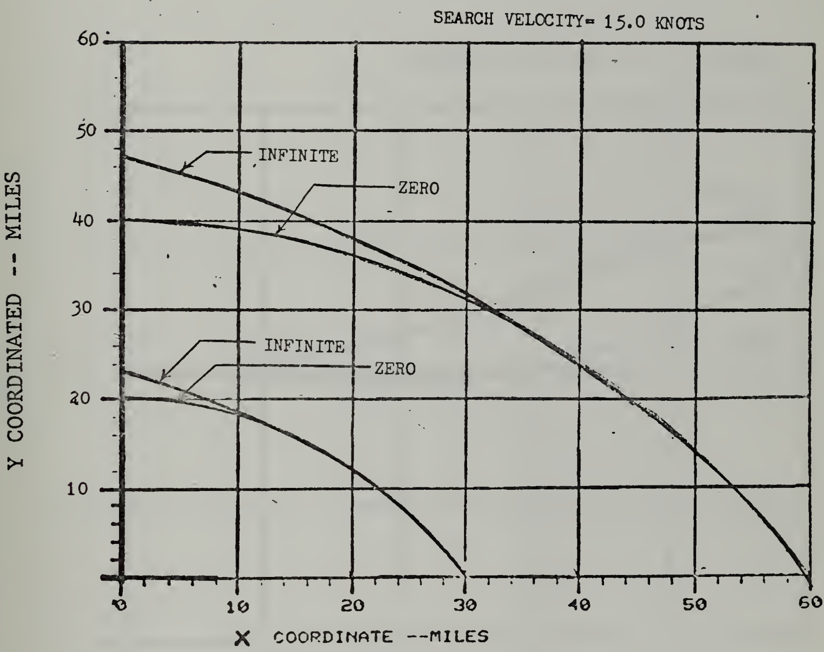


NOTE: For interpolation of control variables:

1. refer to graph 2.B.2 for velocity
2. refer to graph 2.B.3 for heading

GRAPH 1.B

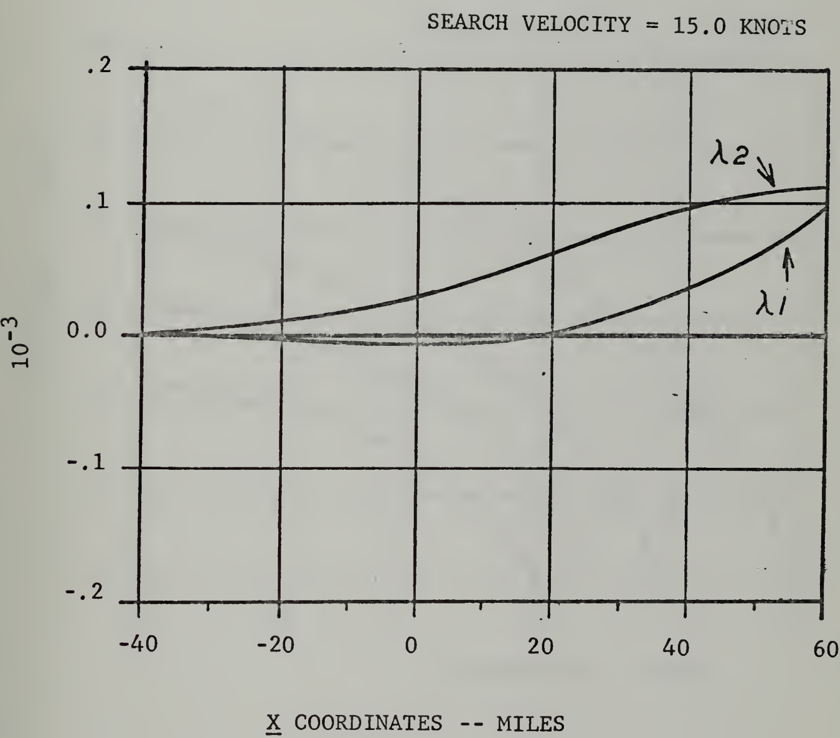
INFINITE AND ZERO
INTEGRATION TIME EVASION TRAJECTORIES



GRAPH 1.C

INFINITE INTEGRATION TIME

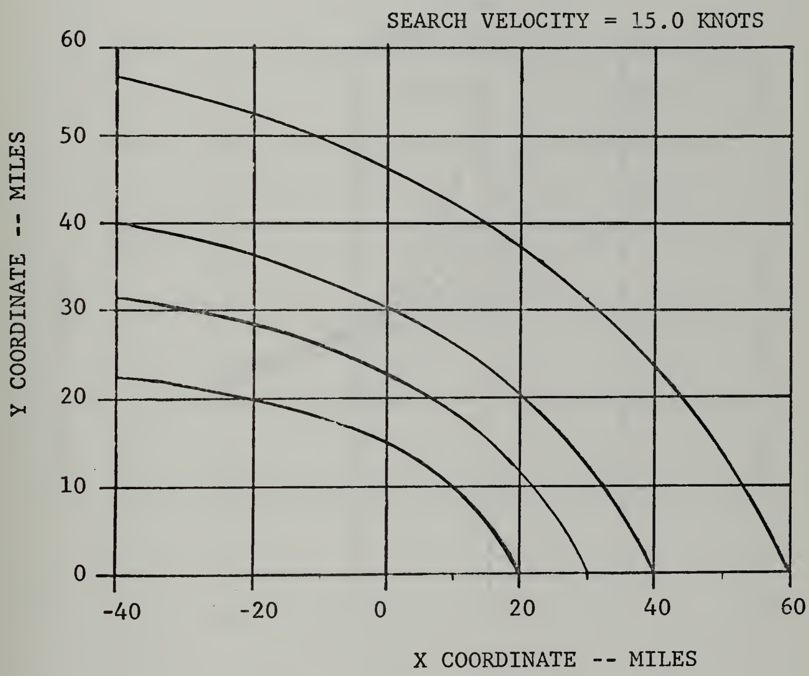
COSTATE VARIABLES VERSUS X COORDINATE



GRAPH 1.D

INFINITE INTEGRATION TIME

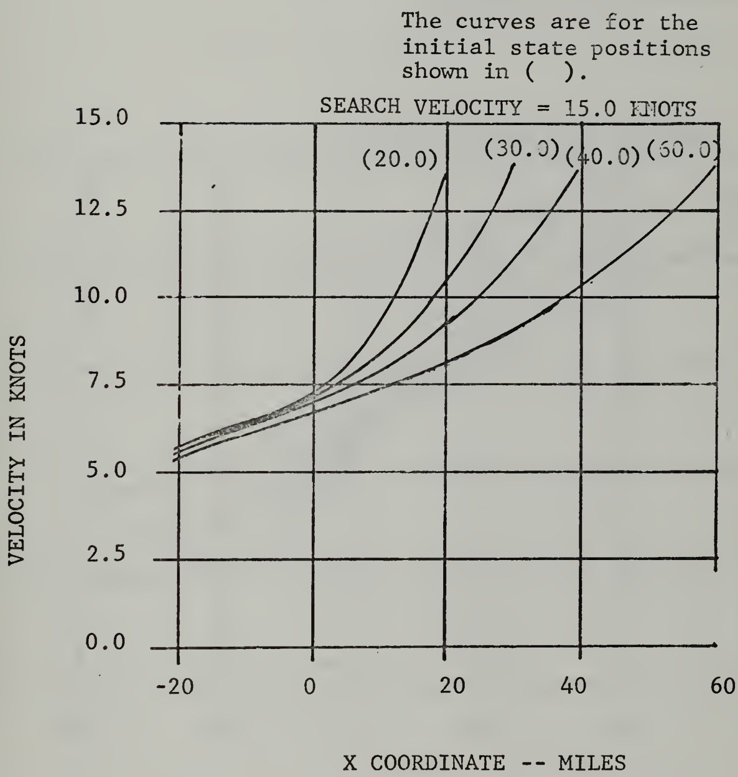
RELATIVE-MOVEMENT PLOT



GRAPH 2.A.1

INFINITE INTEGRATION TIME

VELOCITY VERSUS X COORDINATE

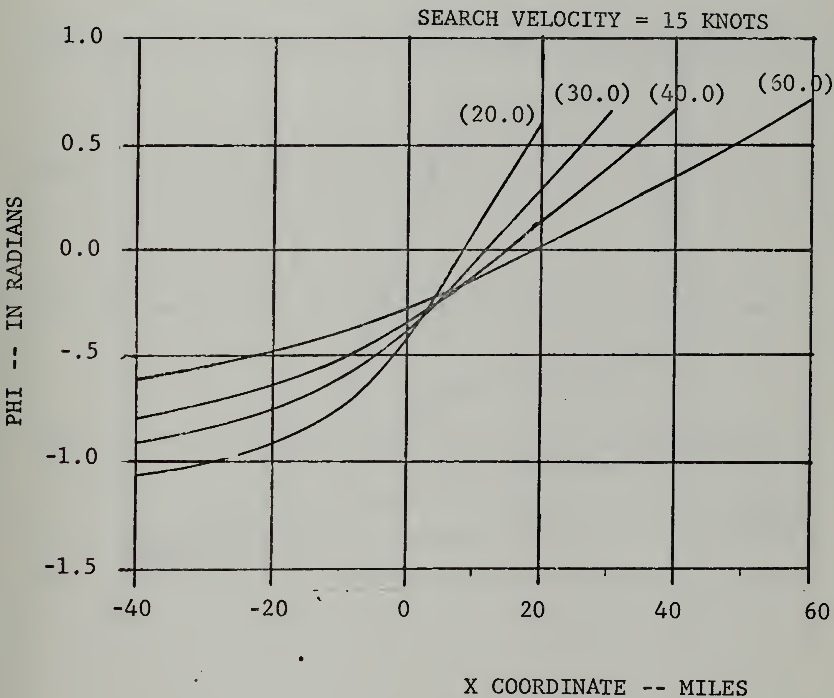


GRAPH 2.A.2

INFINITE INTEGRATION TIME

COURSE HEADING VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).

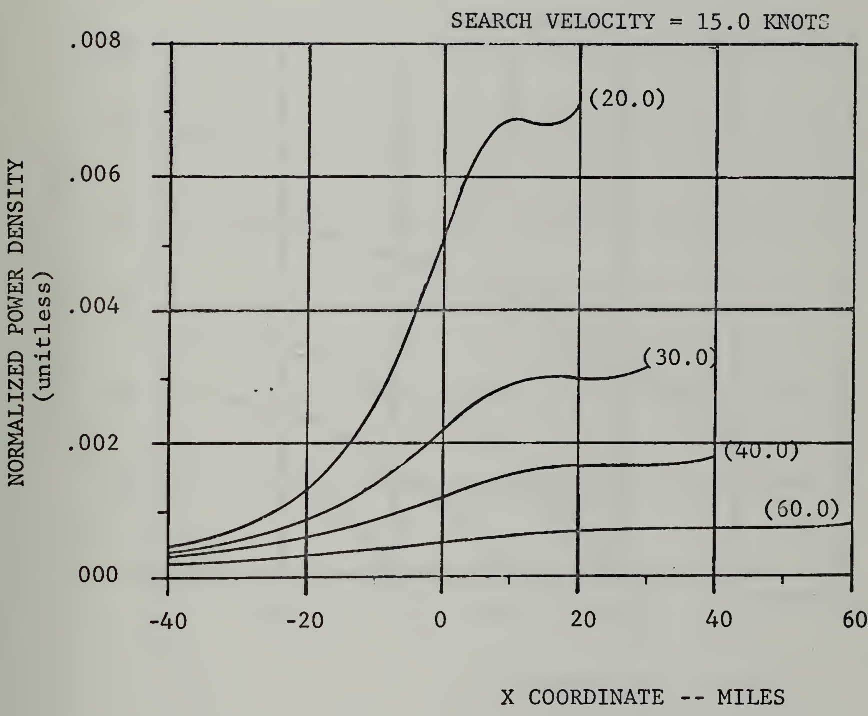


GRAPH 2.A.3

INFINITE INTEGRATION TIME

POWER DENSITY RECEIVED BY SEARCHER
VERSUS X COORDINATE

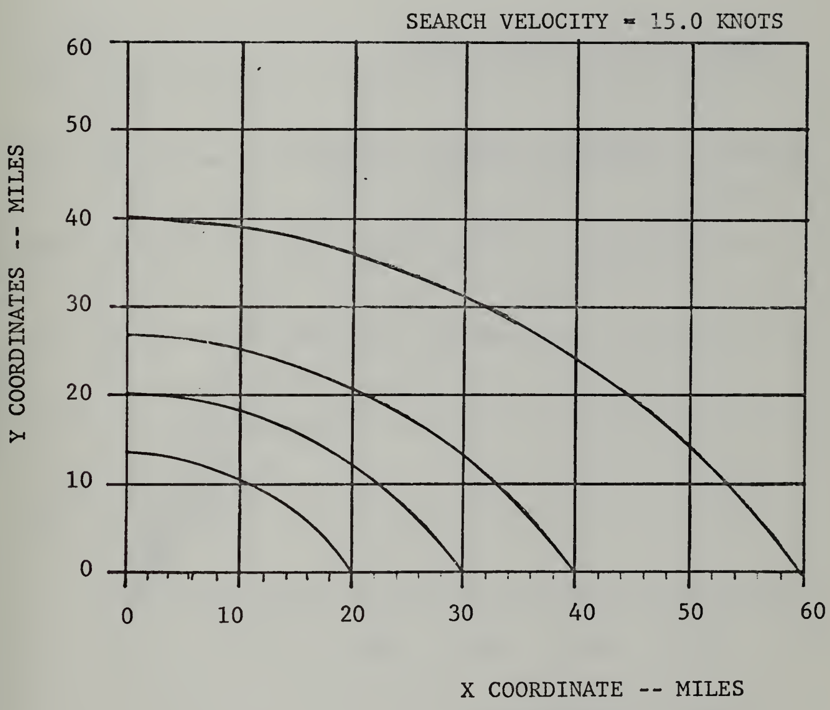
The curves are for initial state positions shown in ( ).



GRAPH 2.A.4

ZERO INTEGRATION TIME

RELATIVE MOVEMENT PLOT

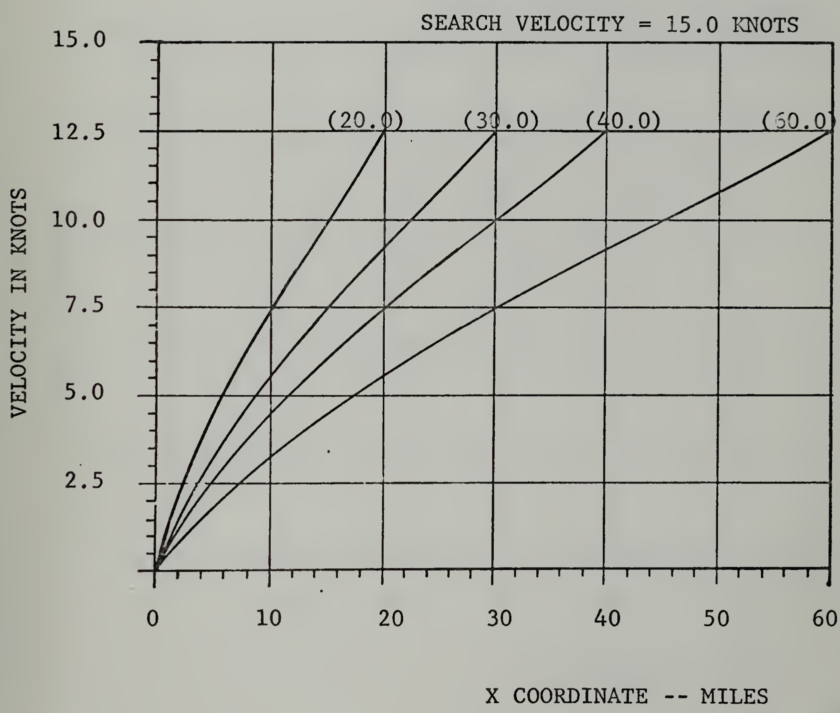


GRAPH 2.B.1

ZERO INTEGRATION TIME

VELOCITY VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).

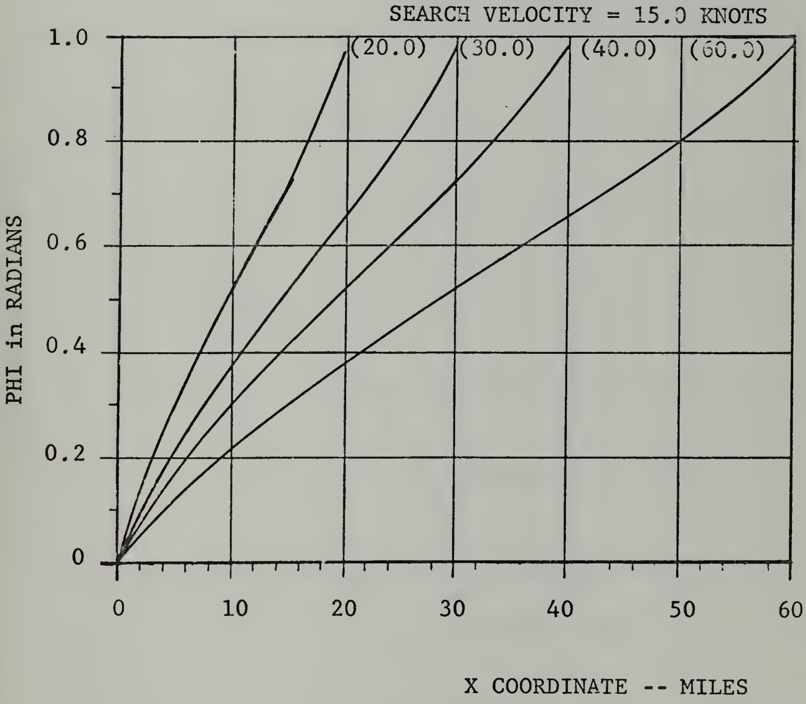


GRAPH 2.B.2

ZERO INTEGRATION TIME

HEADING VERSUS X COORDINATES

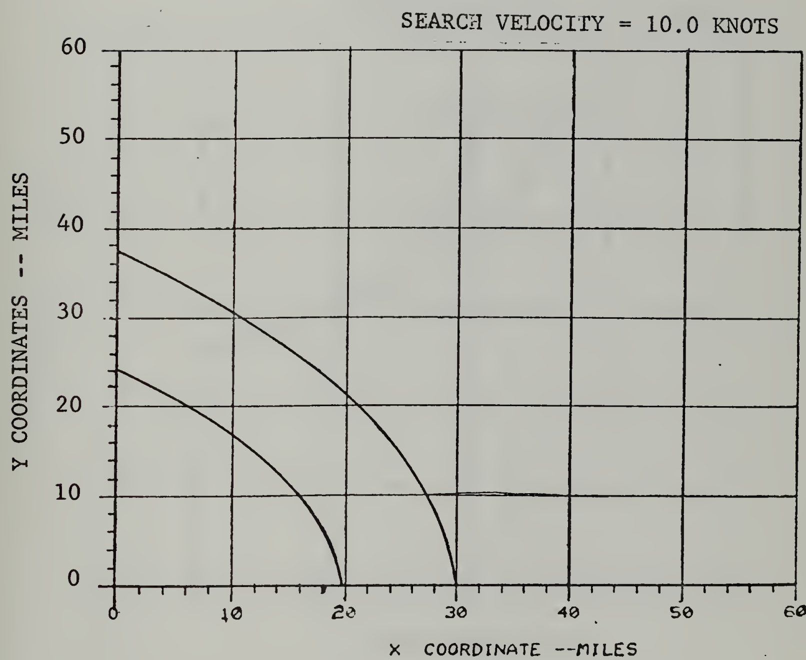
The curves are for the initial state positions shown in ( ).



GRAPH 2.B.3

INFINITE INTEGRATION TIME

RELATIVE MOVEMENT PLOT



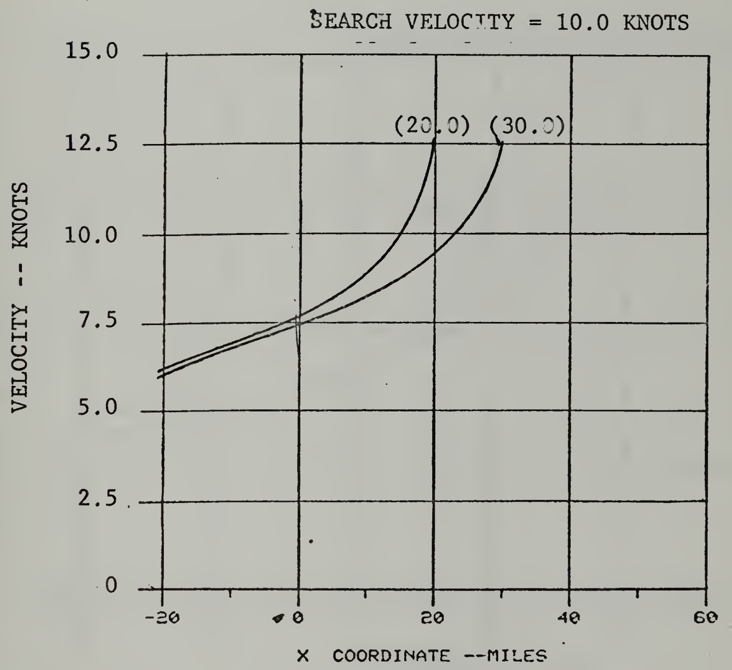
NOTE: Control variables may be obtained from Graphs 3.A.2 and 3.A.3

GRAPH 3.A.1

INFINITE INTEGRATION TIME

EVASION VELOCITY VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).

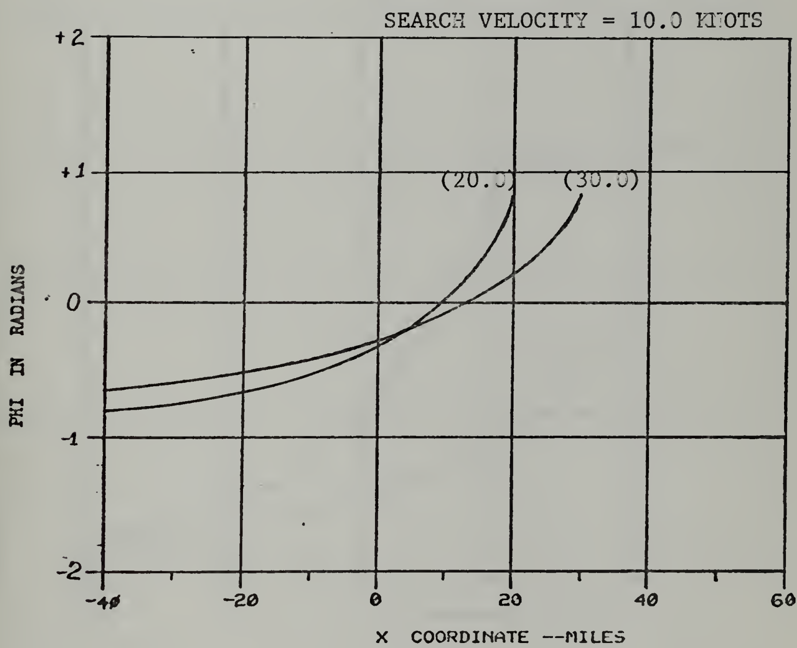


GRAPH 3.A.2

INFINITE INTEGRATION TIME

COURSE HEADING VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).

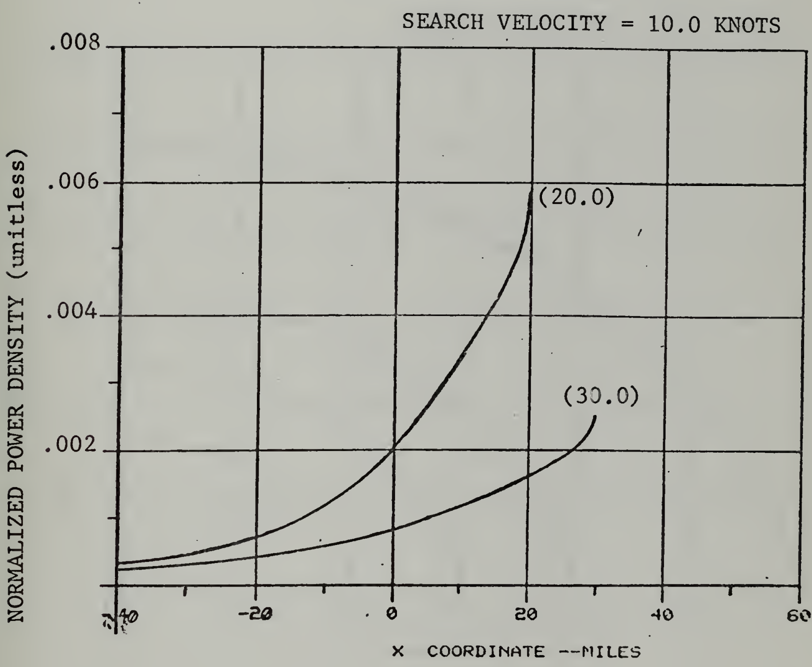


GRAPH 3.A.3

INFINITE INTEGRATION TIME

POWER DENSITY RECEIVED BY SEARCHER VERSUS X COORDINATE

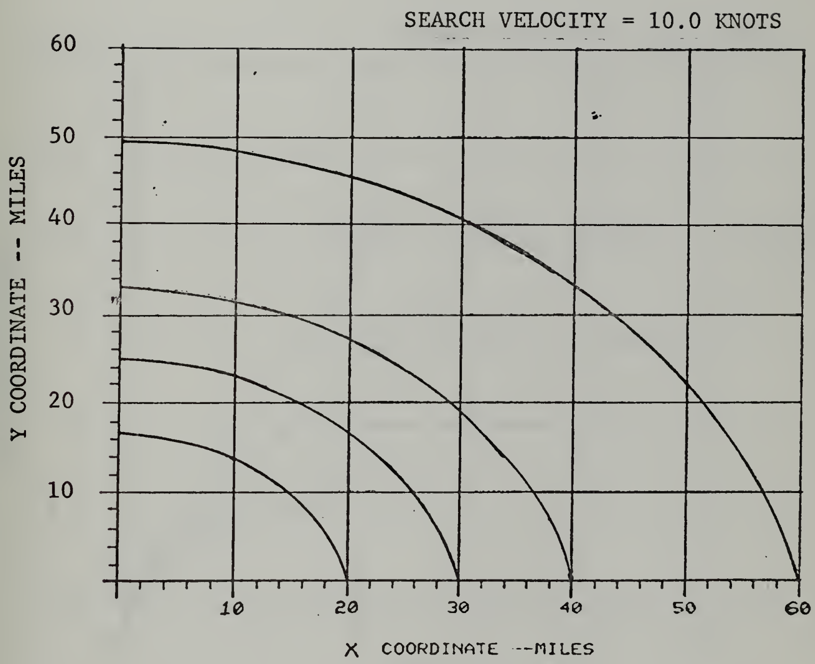
The curves are for the initial state positions shown in ( ).



GRAPH 3.A.4

ZERO INTEGRATION TIME

RELATIVE MOVEMENT PLOT



NOTE: Control variables may be obtained from Graphs 3.B.2 and 3.B.3.

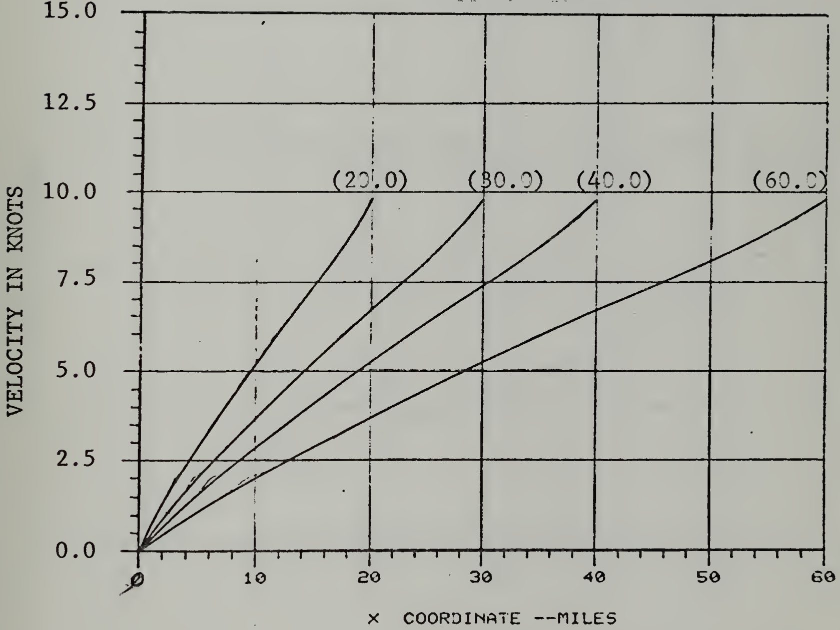
GRAPH 3.B.1

ZERO INTEGRATION TIME

VELOCITY VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).

SEARCH VELOCITY = 10.0 KNOTS

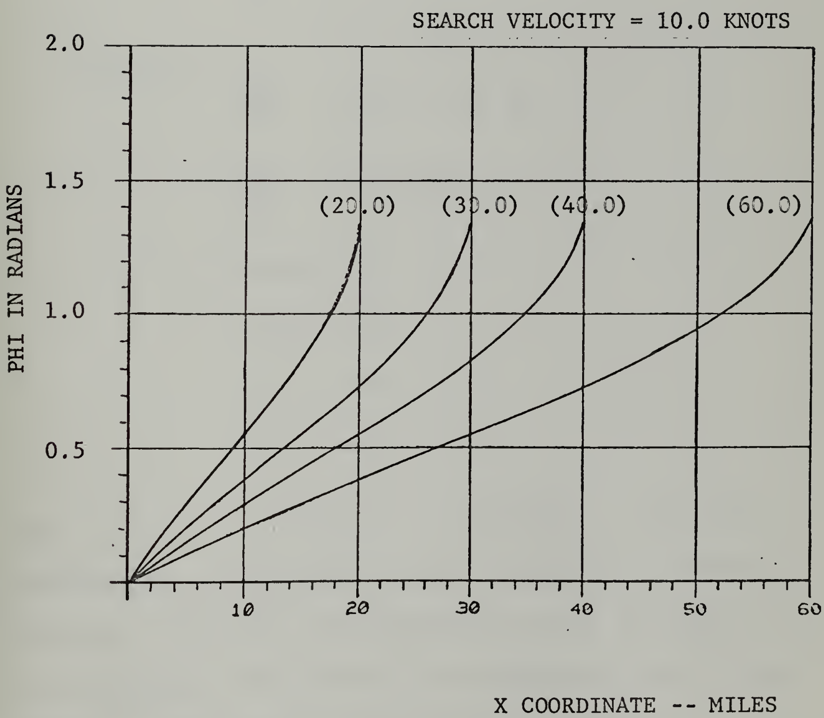


GRAPH 3.B.2

ZERO INTEGRATION TIME

COURSE HEADING VERSUS X COORDINATE

The curves are for the initial state positions shown in ( ).



GRAPH 3.B.3

APPENDIX A

SOLUTION TO INFINITE INTEGRATION TIME PROBLEM

The problem may be stated as,

$$\text{MAX } J = \int_{t_0}^{t_1} \frac{-P(v)}{x^2 + y^2} \, dt$$

subject to;

$$\frac{dx}{dt} = -V_s + v \sin \phi$$

$$\frac{dy}{dt} = v \cos \phi$$

$$x(t_0) = x_0$$

$$y(t_0) = y_0$$

$$x(t_1) = x_1$$

$$y(t_1) = y_1$$

where $P(v) = 1. + 0.00005\, v^4$: Velocity, v, and course heading, ϕ , are functions of time. Velocity is measured in knots. The limits of integration should, in theory, be $\int_0^{\infty}$. As stated $t_0 = 0$ and infinity is approximated with a large number t_1. The symbols are as shown in Figure 1.

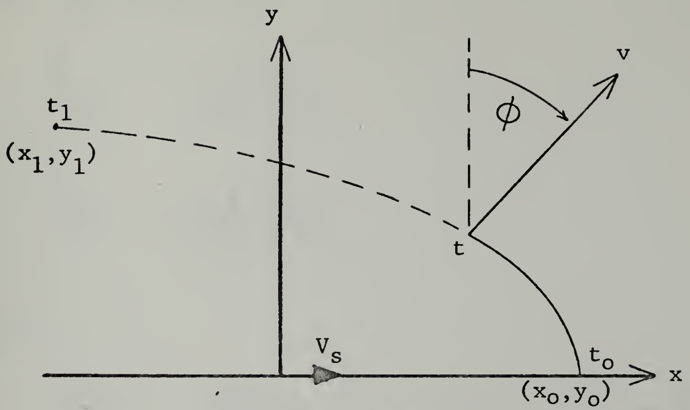


Figure 1

The Hamiltonian function is now defined as

$$H\ (x,y,v,\phi,t) = I\ (x,y,v,\phi,t) + \lambda f\ (x,y,v,\phi,t)\ .$$

In the case under discussion,

$$H(x,y,v,\phi,t) = \frac{-\ P(v)}{x^2 + y^2} + \lambda_1\ (-V_s + v\ \sin\phi\ ) + \lambda_2\ v\ \cos\phi \tag{1}$$

The object is to maximize the Hamiltonian function at every point of time along the optimal trajectory by choice of the control variables, v and ϕ. It is required that;

$$\frac{\partial H}{\partial v} = 0 \quad , \quad \dot{\lambda}_1 = \frac{-\partial H}{\partial x}$$

$$\frac{\partial H}{\partial \phi} = 0 \quad , \quad \dot{\lambda}_2 = \frac{-\partial H}{\partial y} \tag{2}$$

and, $\lambda_1(t_1) = 0 \quad , \quad \lambda_2(t_1) = 0.$

Now, from (1) and (2),

$$\dot{\lambda}_1 = - \frac{2\ x\ P(v)}{(x^2 + y^2)^2} \tag{3}$$

$$\dot{\lambda}_2 = - \frac{2\ y\ P(v)}{(x^2 + y^2)^2} \tag{4}$$

Also,

$$\frac{\partial H}{\partial v} = - \frac{P'(v)}{x^2 + y^2} + \lambda_1 \sin\phi + \lambda_2 \cos\phi \tag{5}$$

$$\frac{\partial H}{\partial \phi} = \lambda_1 v \cos\phi - \lambda_2 v \sin\phi \tag{6}$$

Setting equation (6) equal zero implies,

$$\frac{\lambda_1}{\lambda_2} = \frac{\sin\phi}{\cos\phi}$$

or,

$$\phi = \tan^{-1} (\lambda_1/\lambda_2) \ . \tag{7}$$

From trigonometry and Figure 2 it can be seen that,

$$\cos\phi = \frac{\lambda_2}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}} \tag{8}$$

and

$$\sin\phi = \frac{\lambda_1}{(\lambda_1^2 + \lambda_2^2)^{\frac{1}{2}}} \tag{9}$$

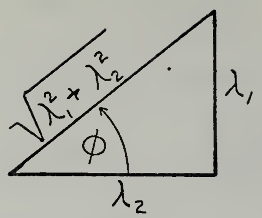


Figure 2

Substitution of equations (8) and (9) into (5) results in;

$$\frac{P'(v)}{x^2+y^2} = \frac{\lambda_1^2+\lambda_2^2}{(\lambda_1^2+\lambda_2^2)^{\frac{1}{2}}} = \sqrt{\lambda_1^2+\lambda_2^2}$$

Therefore, $P'(v) = (x^2+y^2)\sqrt{\lambda_1^2+\lambda_2^2}$ (10)

Since $P(v) = 1.0 + 0.00005\, v^4,$

$$P'(v) = 0.0002\, v^3 \qquad (11)$$

Equating (11) and (10) and solving for v results in

$$v = (5000.\ (x^2+y^2)\sqrt{\lambda_1^2+\lambda_2^2})^{1/3}\ . \qquad (12)$$

Equations (7) and (12) are the conditions for optimality, and must be satisfied at every point in time along the optimal trajectory.

Since $\lambda_1(t_1) = \lambda_2(t_1) = 0$, from (12) it follows that

$$v(t_1) = 0\ . \qquad (13)$$

Since $\phi(t_1) = \tan^{-1}\left(\frac{\lambda_1(t_1)}{\lambda_2(t_1)}\right)$ is an indeterminate form a solution is sought in L'Hopital's rule. Rewriting (7),

$$\tan \phi(t_1) = \frac{\lambda_1(t_1)}{\lambda_2(t_1)}$$

and applying L'Hopital's rule

$$\lim_{t \to t_1} \tan \phi(t_1) = \frac{\frac{\partial \lambda_1(t_1)}{\partial t}}{\frac{\partial \lambda_2(t_1)}{\partial t}} \quad . \tag{14}$$

Substituting (3) and (4) into (14) and canceling similar terms results in

$$\lim_{t \to t_1} \tan \phi(t_1) = \frac{x}{y} \quad . \tag{15}$$

Rewriting (15) the terminal condition for the control variable ϕ is

$$\phi(t_1) = \tan^{-1} \frac{x}{y} \quad . \tag{16}$$

The objective is to find the values of the control variables at every point in time. Of principle interest are the initial velocity and heading. The solution now involves four simultaneous partial differential equations. The

initial conditions of variables x and y could easily be given, but only the terminal conditions are known for λ_1 and λ_2. The solution is to reverse time and call the initial conditions terminal conditions and vice versa. Reversing time changes the signs of the differential equations, but has no effect on the optimal conditions.

The problem solution in backward time may now be stated as,

$$\overset{o}{x} = V_s - v \sin \phi$$

$$\overset{o}{y} = - v \cos \phi$$

$$\overset{o}{\lambda}_1 = \frac{2\ x\ P(v)}{(x^2 + y^2)^2}$$

$$\overset{o}{\lambda}_2 = \frac{2\ y\ P(v)}{(x^2 + y^2)^2}$$

where the "o" represents backward time. The initial conditions are;

$$\left.\begin{aligned} x(t_o) &= \text{parameter} \\ y(t_o) &= \text{parameter} \end{aligned}\right\} \text{state variables}$$

$$\left.\begin{aligned} \lambda_1(t_o) &= 0 \\ \lambda_2(t_o) &= 0 \end{aligned}\right\} \text{costate variables}$$

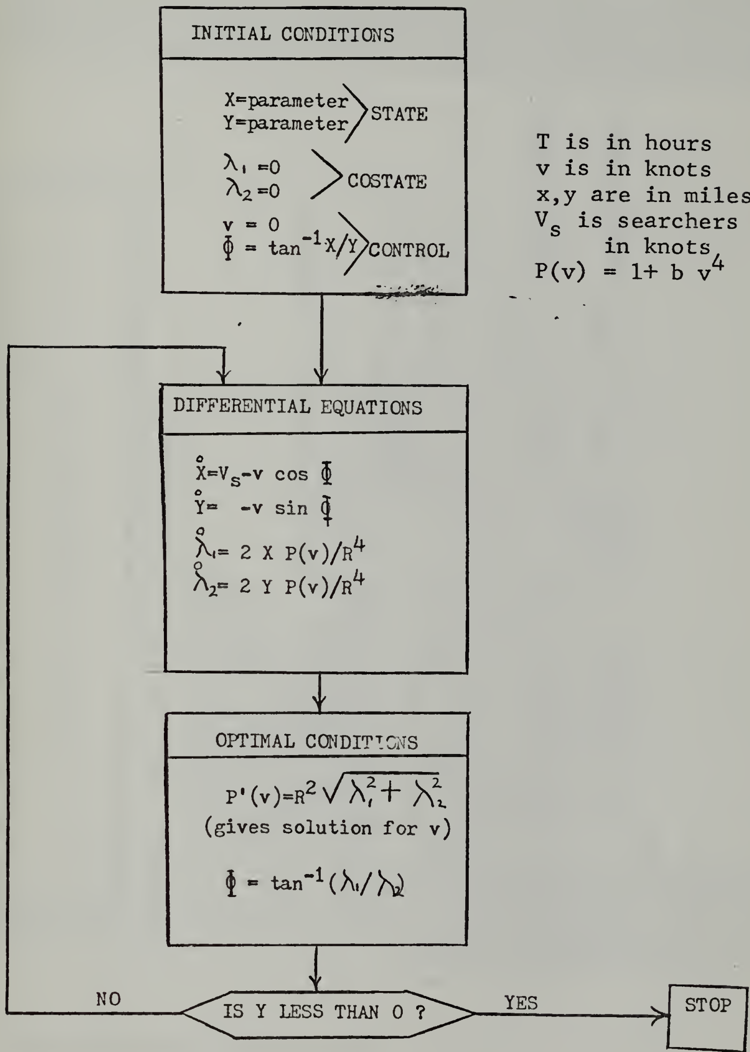
$$\left.\begin{aligned} v(t_o) &= 0 \\ \phi(t_o) &= \tan^{-1}(x/y) \end{aligned}\right\} \text{control variables}$$

APPENDIX B

COMPUTER OUTPUT AND METHODS OF SOLUTION

The computer routines given in this appendix were set to operate on the time sharing system (cp/cms) of an IBM 360 computer. The graphical display package of a Tektronix 4012 display terminal were used. Various termination state positions were attempted until an initial evasion range was selected. Since all the information for a particular trajectory was calculated at each execution, it was only necessary to change the call statements in the plotting package for different visual display.

FLOW CHART FOR INFINITE TIME SOLUTION



T is in hours
v is in knots
x,y are in miles
V_S is searchers speed in knots
$P(v) = 1 + b\ v^4$

```
C       $$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
C       $                                                            $
C       $                                                            $
C       $                                                            $
C       $               INFINITE INTEGRATION TIME                    $
C       $                                                            $
C       $                                                            $
C       $                                                            $
C       $$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
C
C       VARIABLES USED ARE DEFINED AS FOLLOWS
C
C       X.....X COORDINATE IN MILES
C       Y.....Y COORDINATE IN MILES
C       L1......A COSTATE VARIABLE
C       L2......A COSTATE VARIABLE
C       V.....VELOCITY IN KNOTS
C       T.....TIME IN HOURS
C       PHI.....ANGLE OF EVADER'S POSITION MEASURED CLOCKWISE
C             FROM THE Y AXIS
C       P.....POWER RECEIVED BY THE SEARCHER
C       PSUM.....INTEGRATED POWER
C       R.....RANGE OF EVADER FORM SEARCHER IN MILES
C       XPLOT, YPLOT, VPLOT, PPLOT.....TEMPORARY VARIABLES FOR
C                                      SECOND PLOT
      DIMENSION X(401), Y(401), L1(401), L2(401), V(401),T(401),
     1 PHI  (401), P(401), PSUM(401),  R(401)  ,XPLOT(401),
     2    X1(401), Y1(401), X2(401), Y2(401), X3(401), Y3(401)
      REAL L1,L2,L1DOT, L2DOT
      INTEGER*4  I1(6)/'X  C','OORD','INAT','E --','MILE','S   '/
      INTEGER*4  I2(6)/'Y  C','OORD','INAT','E --','MILE','S   '/
      INTEGER*4  I3(7)/'LAMD','A  1','    ','COST','ATE ','VARI','ABLE'/
      INTEGER*4  I4(7)/'LAMD','A  2','    ','COST','ATE ','VARI','ABLE'/
      INTEGER*4  I5(4)/'VELO','CITY','   K','NOTS'/
      INTEGER*4  I6(3)/'TIME','   H','OURS'/
      INTEGER*4  I7(5)/'PHI ','----',' IN ',' RAD','IANS'/
      INTEGER*4  I8(7)/'POWE','R   ','RECE','IVED',' BY ','SEAR','CHER'/
      INTEGER*4  I9(3)/'POWE','R  S','UMED'/
      INTEGER*4  I10(4)/'RANG','E   ','IN M','ILES'/
      DO 4 I=1,401
      X(I)=0.0
      Y(I)=0.0
      L1(I)=0.0
      L2(I)=0.0
      T(I)=0.0
      V(I)=0.0
      P(I)=0.0
      PSUM(I)=0..0
```

```
      PHI  (I)=0.0                                                      STU00490
      R(I)=0.0                                                          STU00500
    4 CONTINUE                                                          STU00510
      PTEMP=0.0                                                         STU00520
      NTIMES=0                                                          STU00530
      X(1)=-40.0                                                        STU00540
      Y(1)=56.7                                                         STU00550
    5 CONTINUE                                                          STU00560
      L1(1)=0.0                                                         STU00570
      L2(1)=0.0                                                         STU00580
      T(1)=0.0                                                          STU00590
      V(1)=0.0                                                          STU00600
      DELTA=0.02                                                        STU00610
      PTEMP=0.0                                                         STU00620
      PHI  (1)=ATAN(X(1)/Y(1))                                          STU00630
      VSERC=15.0                                                        STU00640
C                                                                       STU00650
      DO 30 I=1,400                                                     STU00660
      XDOT=+VSERC-V(I)*SIN( PHI (I))                                    STU00670
      YDOT=-V(I)*COS( PHI (I))                                          STU00680
      TEMP=2.0*(1.0+0.00005*V(I)**4)/(X(1)**2+Y(I)**2)**2               STU00690
      L1DOT=X(I)*TEMP                                                   STU00700
      L2DOT=Y(I)*TEMP                                                   STU00710
      X(I+1)=X(I)+DELTA*XDOT                                            STU00720
      Y(I+1)=Y(I)+DELTA*YDOT                                            STU00730
      L1(I+1)=L1(I)+DELTA*L1DOT                                         STU00740
      L2(I+1)=L2(I)+DELTA*L2DOT                                         STU00750
      T(I+1)=T(I)+DELTA                                                 STU00760
      P(I)=(1.0+0.00005*V(I)**4)/(X(I)**2+Y(I)**2)                      STU00770
      PSUM(I)=PTEMP+P(I)*DELTA                                          STU00780
      PTEMP=PSUM(I)                                                     STU00790
      R(I)=SQRT(X(I)**2+Y(I)**2)                                        STU00800
      IF (Y(I+1).LE.0.0) GO TO 40                                       STU00810
      V(I+1)=(5000.0*(X(I+1)**2+Y(I+1)**2)*SQRT(L1(I+1)**2+L2(I+1)**2)) STU00820
     1**.33333                                                          STU00830
      RATIO=L1(I+1)/L2(I+1)                                             STU00840
       PHI (I+1)=ATAN(RATIO)                                            STU00850
       IVALUE = I+1                                                     STU00860
   30 CONTINUE                                                          STU00870
C                                                                       STU00880
   40 CONTINUE                                                          STU00890
   48 NTIMES=NTIMES +1                                                  STU00900
      IF (NTIMES.EQ.1) GO TO 50                                         STU00910
      IF (NTIMES.EQ.2) GO TO 52                                         STU00920
      IF (NTIMES.EQ.3) GO TO 54                                         STU00930
      IF (NTIMES.EQ.4) GO TO 60                                         STU00940
   50   DO 51 I=1,401                                                   STU00950
      X1(I)=X(I)                                                        STU00960
```

```
      Y1(I)=L2(I)
51     CONTINUE
      X(1)=-40.0
      Y(1)=56.7
      IX1=IVALUE
      GO TO 60
52      DO 53 I=1,401
      X2(I)=X(I)
      Y2(I)=Y(I)
53     CONTINUE
      X(1)=-40.0
      Y(1)=40.2
      IX2=IVALUE
      GO TO 6
54     DO 55 I=1,401
      X3(I)=X(I)
      Y3(I)=Y(I)
55     CONTINUE
      X(1)=-40.0
      Y(1)=56.7
      IX3=IVALUE
      GO TO 6
6      DO 7 I=2,401
      X(I)=0.0
      Y(I)=0.0
      L1(I)=0.0
      L2(I)=0.0
      T(I)=0.0
      V(I)=0.0
      P(I)=0.0
      PSUM(I)=0.0
       PHI (I)=0.0
      R(I)=0.0
7     CONTINUE
      GO TO 5
60    CONTINUE
      CALL INIT
      CALL STEPL  (4)
      CALL DLIMX(-40.0,60.0)
      CALL DLIMY (-0.0002,0.0002)
      CALL SLIMX (150,946)
      CALL SLIMY (100,740)
      CALL NPTS (IVALUE)
      CALL PLOT (X,L1)
       CALL PAUSE
      CALL NPTS (IX1)
      CALL CPLOT (X1,Y1)
      CALL PAUSE
```

```
      GO TO 90
80    CALL NPTS (IX2)
      CALL CPLOT (X2,Y2)
      CALL PAUSE
      CALL NPTS (IX3)
      CALL CPLOT (X3,Y3)
      CALL PAUSE
      CALL TITLE(24,I1)
      CALL TITLE(-28,I3)
      CALL PAUSE
90    CALL FIN
      WRITE (6,66) IVALUE, V(IVALUE)
66    FORMAT (1X,'VEL REF SUB ',I3,' = ',F6.2)
      STOP
      END
```

```
C                    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
C                    X                              X
C                    X                              X
C                    X                              X
C                    X                              X
C                    X     ZERO INTEGRATION TIME    X
C                    X                              X
C                    X                              X
C                    XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX
      DIMENSION R(401), PSI(401), PHI(401), V(401), P(401),
     1  RHO(401), U(401), X(401), Y(401),
     2   X1(401), Y1(401), X2(401), Y2(401), X3(401), Y3(401)
      INTEGER*4 I1(6)/'X  C','OORD','INAT','E --','MILE','S   '/
      INTEGER*4 I2(6)/'Y  C','OORD','INAT','E --','MILE','S   '/
      INTEGER*4 I3(7)/'LAMD','A  1','    ','COST','ATE ','VARI','ABLE'/
      INTEGER*4 I4(7)/'LAMD','A  2','    ','COST','ATE ','VARI','ABLE'/
      INTEGER*4 I5(4)/'VELO','CITY','   K','NOTS'/
      INTEGER*4 I6(3)/'TIME','   H','OURS'/
      INTEGER*4 I7(5)/'PHI ','----',' IN ',' RAD','IANS'/
      INTEGER*4 I8(7)/'POWE','R   ','RECE','IVED',' BY ','SEAR','CHER'/
      INTEGER*4 I9(3)/'POWE','R  S','UMED'/
      INTEGER*4 I10(4)/'RANG','E   ','IN M','ILES'/
      INTEGER*4 I11(4)/' PSI',' IN ',' RAD','IANS'/
      FUNPHI(A,A3,A5)=A-A3*A**3-A5*A**5
       CALL ERRSET (253,0,-1,0,0,0)
       CALL ERRSET (257,0,-1,0,0,0)
      DO 10 I=1,401
      R(I)=0.0
      V(I)=0.0
      P(I)=0.0
      PSI(I)=0.0
      PHI(I)=0.0
      U(I)=0.0
      RHO(I)=0.0
      X(I)=0.0
      Y(I)=0.0
   10 CONTINUE
      VSERC=11.0
      NTIMES=0
      BETA=0.0000625*VSERC**4
      A3=2.0*BETA
      A5=-2.0*BETA*(12.0*BETA+1.0)
      DELPSI =0.01
      R0=13.44
C
    5    CONTINUE
C
      Y(1)=R0
```

```
      R(1)=RO                                                           CLY00490
      U(1)=1.0                                                          CLY00500
      PFINAL =1.0/RO**2                                                 CLY00510
      DO 30 I=1,400                                                     CLY00520
      IF (PSI(I).GE.1.57) GO TO 40                                      CLY00530
      IF (PSI(I).GT.0.115) GO TO 20                                     CLY00540
      PSI (I+1) =PSI (I) +0.02                                          CLY00550
      A=PSI(I+1)                                                        CLY00560
      PHI(I+1)=FUNPHI (A,A3,A5)                                         CLY00570
      RHO (I+1) =ARSIN (PHI(I+1))                                       CLY00580
      U(I+1)=1.0+BETA*RHO(I+1)**4                                       CLY00590
      R(I+1)=SQRT(U(I+1)*RO**2)                                         CLY00600
      P(I+1)=(1.0+0.00005*V(I+1)**4)/R(I+1)**2                          CLY00610
      V(I+1)=(20000.0*(R(I+1)**2*PFINAL-1.0))**0.25                     CLY00620
      X(I+1)=R(I+1)*SIN(PSI(I+1))                                       CLY00630
      Y(I+1)=R(I+1)*COS(PSI(I+1))                                       CLY00640
      GO TO 30                                                          CLY00650
20    DELU=2*U(I)*ATAN(PSI(I)-PHI(I))                                   CLY00660
      U(I+1)=U(I)+DELU*DELPSI                                           CLY00670
      PSI(I+1)=PSI(I)+DELPSI                                            CLY00680
      R(I+1)=SQRT(U(I+1)*RO**2)                                         CLY00690
      V(I+1)=(20000.0*(R(I+1)**2*PFINAL-1.0))**0.25                     CLY00700
      P(I+1)=(1.0+0.00005*V(I+1)**4)/R(I+1)**2                          CLY00710
      PHI(I+1)=ARSIN(V(I+1)/VSERC)                                      CLY00720
      RHO(I+1)=V(I+1)/VSERC                                             CLY00730
      X(I+1)=R(I+1)*SIN(PSI(I+1))                                       CLY00740
      Y(I+1)=R(I+1)*COS(PSI(I+1))                                       CLY00750
       IVALUE = I+1                                                     CLY00760
30       CONTINUE                                                       CLY00770
C                                                                       CLY00780
40    CONTINUE                                                          CLY00790
48    NTIMES=NTIMES +1                                                  CLY00800
      IF (NTIMES.EQ.1) GO TO 50                                         CLY00810
      IF (NTIMES.EQ.2) GO TO 52                                         CLY00820
      IF (NTIMES.EQ.3) GO TO 54                                         CLY00830
      IF (NTIMES.EQ.4) GO TO 60                                         CLY00840
50      DO 51 I=1,401                                                   CLY00850
      X1(I)=X(I)                                                        CLY00860
      Y1(I)=PHI(I)                                                      CLY00870
51     CONTINUE                                                         CLY00880
      RO=20.14                                                          CLY00890
      IX1=IVALUE                                                        CLY00900
      GO TO 6                                                           CLY00910
52      DO 53 I=1,401                                                   CLY00920
      X2(I)=X(I)                                                        CLY00930
      Y2(I)=PHI(I)                                                      CLY00940
53     CONTINUE                                                         CLY00950
      RO=26.88                                                          CLY00960
```

```
      IX2=IVALUE
      GO TO 6
54     DO 55 I=1,401
      X3(I)=X(I)
      Y3(I)=PHI(I)
55     CONTINUE
      R0=40.26
      IX3=IVALUE
      GO TO 6
6      DO 7 I=1,401
      X(I)=0.0
      Y(I)=0.0
      U(I)=0.0
      RHO(I)=0.0
      PSI(I)=0.0
      V(I)=0.0
      P(I)=0.0
       PHI (I)=0.0
      R(I)=0.0
7     CONTINUE
      GO TO 5
60    CONTINUE
      CALL INIT
      CALL STEPL  (4)
      CALL DLIMX(0.0,60.0)
      CALL DLIMY (0.0,2.0)
      CALL SLIMX (150,946)
      CALL SLIMY (100,740)
      CALL NPTS (IVALUE)
      CALL PLOT (X,PHI)
       CALL PAUSE
      CALL NPTS (IX1)
      CALL CPLOT (X1,Y1)
      CALL PAUSE
      CALL NPTS (IX2)
      CALL CPLOT (X2,Y2)
      CALL PAUSE
      CALL NPTS (IX3)
      CALL CPLOT (X3,Y3)
      CALL PAUSE
      CALL TITLE(24,I1)
      CALL TITLE(-20,I7)
      CALL PAUSE
      CALL FIN
      WRITE (6,70) X(IVALUE), X1(IX1), X2(IX2), X3(IX3), NTIMES
70    FORMAT (////,1X,'THE TERMINAL X VALUES ARE ',  4F15.3,I4)
      STOP
      END
```

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